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an involute with respect to the center of its circle is a spiral tractrix, that is, a curve which, in polar coördinates, has a tangent of constant length (Haton de la Goupillière,[4] 1863)—The centers of curvature for the points of contact of an involute rolling on a straight line is a parabola (Cesàro,[1] 1884)—The points of contact of tangents drawn to an involute from any point of its plane lie on a limaçon of Pascal (Fouret,[2] 1888).

The involute of a circle seems to have been first conceived in 1693 when Huygens was considering clocks without pendulums which might be of service on sea going vessels.[3] In this connection he originated an apparatus in which the involute of a circle plays an essential rôle.

In 1891 it became desirable to install in the Royal Observatory at Greenwich a larger telescope. This necessitated that a larger dome, 36 feet in diameter, be built upon a circular wall 31 feet 4 inches in diameter. The form adopted was that of a "surface generated by the revolution of an involute of a circle, 7 feet in diameter, with its center in the plane of the rail, and 5 feet from the axis, the curve being completed near the apex by an arc of a circle (of 13 feet 3 inches radius) so that it cuts the axis at right angles. The diameter of the dome is 36 feet at a height of 7 feet above the rail."[4] ARC.

SOLUTIONS.

2809 [1920, 80]. Proposed by the late L. G. WELD.

Find the nth term of the series defined by the relation, $u_{i+2} = u_i + u_{i+1}$, in which $u_1 = u_2 = 1$.

SOLUTION BY HENRI SEBBAN, Boufarik, Algeria.

Consider the sequence

$$1, 1, 2, 3, 5, 8, \cdots$$

in which each term beginning with the third is the sum of the two terms immediately preceding. In order to express a term as a function of u_1, u_2 and i, let us set $u_i = Ax_1^i + Bx_2^i$ and endeavor to determine A, B, x_1 and x_2 so as to satisfy the law of the sequence. Then

$$\begin{aligned} Ax_1^i + Bx_2^i &= Ax_1^{i-1} + Bx_2^{i-1} + Ax_1^{i-2} + Bx_2^{i-2} \\ &= Ax_1^{i-2}(x_1 + 1) + Bx_2^{i-2}(x_2 + 1), \end{aligned}$$

and from this it follows that $x_1 + 1 = x_1^2$, $x_2 + 1 = x_2^2$. Hence x_1 and x_2 are the two roots of the equation $x^2 - x - 1 = 0$, for we cannot have $x_1 = x_2$ unless the sequence reduces to a geometric progression, $u_i = Kx^i$ of ratio x which is not the case when $u_1 = u_2 = 1$. It will be seen that in the general case A and B are determined by the initial conditions and are different from zero. Hence we may set

$$x_1 = \frac{1 + \sqrt{5}}{2}, \quad x_2 = \frac{1 - \sqrt{5}}{2}, \quad \text{and} \quad u_i = A\left(\frac{1 + \sqrt{5}}{2}\right)^i + B\left(\frac{1 - \sqrt{5}}{2}\right)^i,$$

where A and B are to be determined from the conditions $Ax_1 + Bx_2 = 1$, $Ax_1^2 + Bx_2^2 = 1$. Hence, we have after certain reductions

$$A = -B = \frac{1}{\sqrt{5}} \quad \text{and} \quad u_i = \frac{(1 + \sqrt{5})^i - (1 - \sqrt{5})^i}{2^i\sqrt{5}}.$$

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[1] *Mathesis*, 1884, pp. 233–235.

[2] *Journal de Mathématiques Spéciales*, 1888, p. 261.

[3] *Œuvres Complètes de Christian Huygens*, volume 10, 1905, pp. 514–515.

[4] *Monthly Notices of the Royal Astronomical Society*, volume 51, May 8, 1891, p. 436.

It will be found that for this series the following relations are true:

$$u_{i+2}u_{i-1} - u_i u_{i+1} = \pm 1, \qquad u_i^4 - u_{i-2}u_{i-1}u_{i+1}u_{i+2} = 1, \qquad \text{and} \qquad S_i = u_{i+2} - u_2,$$

where S_i is the sum of the first i terms of the series. This latter series has received the names Fibonacci's and Lamé's, and is the subject of several interesting studies. In the *Journal de Mathématiques Elémentaires*(Vuibert), 44e année, p. 11, it is shown that the sum of the squares of two consecutive terms of Fibonacci's series is a term of the series. In the *Intermédiaire des Mathématiciens*, 1916, pages 64 and 599, it is proved that if p is a prime of the form $10n \pm 3$ it is a divisor of u_{p+1}; if of the form $10n \pm 1$ it is a divisor of u_{p-1}. In *Nouvelles Annales de Mathématiques*, 1884, p. 533, is a proof of the theorem that $(2u + 1)^i - u^{3i} = 1$ provided that the exponents in the expansion are replaced by indices and the resulting terms u_n are terms of Fibonacci's series.[1]

Also solved by NORMAN ANNING, J. P. BALLANTINE, E. B. ESCOTT, H. L. OLSON, E. J. OGLESBY, and LOUIS WEISNER.

2811 [1920, 81]. Proposed by J. L. RILEY, Stephenville, Texas.

Given cube roots of 60, 61, 63 and 64, to find the cube root of 62 by the method of differences.

SOLUTION AND REMARKS BY E. B. ESCOTT, Oak Park, Illinois.

Let u_1, u_2, u_3, u_4, u_5 be a series of functions with equidistant arguments. By the method of differences, the first term of the fourth order of difference of u_1 is

$$u_5 - 4u_4 + 6u_3 - 4u_2 + u_1.$$

Let this equal zero and solve for u_3, whence

$$u_3 = \tfrac{1}{6}[4(u_2 + u_4) - (u_1 + u_5)].$$

If

$$u_1 = \sqrt[3]{60} = 3.91486\ 76412 -, \qquad u_2 = \sqrt[3]{61} = 3.93649\ 71831, \qquad u_3 = \sqrt[3]{62},$$
$$u_4 = \sqrt[3]{63} = 3.97905\ 72079 - \qquad \text{and} \qquad u_5 = \sqrt[3]{64} = 4,$$

we have by above formula, $\sqrt[3]{62} = 3.95789\ 16538$. This is true to 7 decimals, the value to 10 decimals being $3.95789\ 16097 -$.

A more general method is to use Newton's Interpolation Formula, where the intervals are not necessarily equal, *i.e.*,

$$u_x = u_a + (x - a)\cdot\delta'(a, b) + (x - a)(x - b)\cdot\delta''(a, b, c) + \cdots, \tag{1}$$

where

$$\delta'(a, b) = \frac{u_a - u_b}{a - b}, \qquad \delta''(a, b, c) = \frac{\delta'(a, b) - \delta'(b, c)}{a - c}, \qquad \cdots.$$

See Thiele, *Interpolationsrechnung*, page 5, or Boole, *Finite Differences*, Chapter 3, Exercise 19. In this problem, $u_{62} = \sqrt[3]{62}$, $u_a = u_{60} = \sqrt[3]{60}$, $u_b = u_{61} = \sqrt[3]{61}$, $u_c = u_{63} = \sqrt[3]{63}$, $u_d = u_{64} = \sqrt[3]{64}$.

Then

$$\delta'(61, 60) = \frac{\sqrt[3]{61} - \sqrt[3]{60}}{61 - 60} = .02162\ 95419,$$

$$\delta'(63, 61) = \frac{\sqrt[3]{63} - \sqrt[3]{61}}{63 - 61} = .02128\ 00124,$$

$$\delta'(64, 63) = \frac{\sqrt[3]{64} - \sqrt[3]{63}}{64 - 63} = .02094\ 27921,$$

$$\delta''(63, 61, 60) = \frac{\delta'(63, 61) - \delta'(61, 60)}{63 - 60} = -\ .00011\ 65098,$$

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[1] The literature of this series has been discussed already in this MONTHLY, *1918*, 234–238, 462, by R. C. Archibald; reference to a recent extended elaboration of these notes was made *1920*, 314.—EDITORS.